Math 31 - Homework 5

Due Friday, August 3

Easy

1. Let $G = \langle a \rangle$ be a cyclic group of order n. You showed on a previous homework that if $m \mid n$, then G has a unique subgroup of order m, namely

$$N = \langle a^{n/m} \rangle.$$

Moreover, $N \lhd G$ since G is abelian.

(a) How many cosets of N are there in G? Find them.

(b) Given two cosets of N, what is their product? Verify that each coset is a power of the coset Na.

(c) Conclude that G/N is a cyclic group. What is its order?

2. [Herstein, Section 2.6 #2] Recall that $\mathbb{R}^\times$ is the group of nonzero real numbers (under multiplication), and let $N = \{-1, 1\}$. Show that N is a normal subgroup of $\mathbb{R}^\times$, and that $\mathbb{R}^\times/N$ is isomorphic to the group of positive real numbers under multiplication. [**Hint:** Use the First Homomorphism Theorem.]

3. [Herstein, Section 3.2 #2] Find the cycle decomposition and order of each of the following permutations.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix}$

4. [Herstein, Section 3.3 #1] Determine whether each permutation is even or odd.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix}$

(b) $(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)(7\ \ 8\ \ 9)$

(c) $(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 7)$

(d) $(1\ \ 2)(1\ \ 2\ \ 3)(4\ \ 5)(5\ \ 6\ \ 8)(1\ \ 7\ \ 9)$

5. [Herstein, Section 3.3 #5] Suppose you are told that the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & & & 7 & 8 & 9 & 6 \end{pmatrix}$$

in S_9, where the images of 4 and 5 have been lost, is an even permutation. What must the images of 4 and 5 be?

Medium

6. Prove that if G is abelian and $N \leq G$, then G/N is abelian. [**Hint:** You may want to use a result from the last homework assignment.]

7. If G is a group and $M \lhd G$, $N \lhd G$, prove that $M \cap N \lhd G$. [You proved on an earlier assignment that $M \cap N$ is a subgroup of G, so you only need to prove that it is normal.]

8. Let G be a group. Recall from a previous homework that the **center** of G is the set $Z(G)$ defined by

$$Z(G) = \{x \in G : xa = ax \text{ for all } a \in G\}.$$

You proved that $Z(G)$ is a subgroup of G.

(a) Prove that $Z(G) \lhd G$.

(b) If $G/Z(G)$ is cyclic, prove that G is abelian.

9. Let G be a group and $H \leq G$. If $[G : H] = 2$, prove that H is normal in G.

Hard

10. Let G be a group, and recall that $\mathrm{Aut}(G)$ is the set of all automorphisms of G. You proved on the last homework that $\mathrm{Aut}(G)$ forms a group under composition.

(a) Given $g \in G$, define a function $\theta_g : G \to G$ by

$$\theta_g(a) = gag^{-1}$$

for all $a \in G$. Show that $\theta_g \in \mathrm{Aut}(G)$. (Such an automorphism is called an **inner automorphism**.)

(b) Let $\mathrm{Inn}(G)$ denote the set of all inner automorphisms of G. Then $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ by part (a). Show that $\mathrm{Inn}(G)$ is actually a subgroup of $\mathrm{Aut}(G)$.

(c) Prove that $\mathrm{Inn}(G)$ is a normal subgroup.